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DIRECT ELECTRIC CURRENT IN A MEDIUM WITH A LARGE NUMBER OF CRACKS

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UDC 539.3

The solution of the problem of an electrostatic field and a field of direct electric current in a medium containing a large number of randomly arranged cracks is the theoretical basis for some important methods of nondestructive checking of damage to materials (e.g., metals, rocks) [1, 2].

In the present study we propose an approach for finding the multipoint moments of a statistical solution of this problem. For the case of a direct electric current we consider a method of constructing the means (mathematical expectations) and correlation functions of the current and voltage vector fields with an electric field in a medium with cracks.

In recent years there have been a great many studies devoted to the description of the effective thermal, electrical, and magnetic properties of inhomogeneous materials (see, e.g., [3]). The problem of constructing the effective parameters of an inhomogeneous medium reduces to calculating the mean value of the random field of the solution. For some stochastic structures this problem admits of fairly good approximate, or even exact, solutions (exact summation of series of perturbation theory). However, in calculating the variance of the solution, when we construct the correlation functions in terms of which we can express, e.g., the mean energy density of the field, visible results can be obtained only for the case of weak inhomogeneity, where we confine ourselves to the first terms of the series of perturbation theory [4].

In the specific case of a medium containing a field of isolated inhomogeneities, a number of authors have used the effective (self-consistent) field method, which is well known in many-particle theory.

It should be noted that the idea of self-consistency for describing the effective properties of an inhomogenous medium can be used in various forms. In [5, 6] self-consistent solutions of an electrical conduction problem were constructed on the basis of the assumption that each typical inhomogeneity – for example, a polycrystal grain – behaves as if it were isolated in a homogeneous medium whose properties coincide with the effective properties of the entire medium, while the field in which such an inhomogeneity is situated was taken to be equal to the external field. Such a modification of the method is sometimes called the effective-medium method [7].

In the present study, in constructing a self-consistent solution for a medium containing plane elliptical cracks, it is assumed that each crack behaves as if it were isolated in a principal medium with known properties, and the presence of the surrounding cracks is taken into account by means of the effective field in which it is situated. Unlike the usual formulations of the method, in which the effective field is chosen to be the same for all particles [8], here we assume that this field is random, varying from crack to crack. To construct the equations which will be satisfied by the statistical moments of the effective field, we make use of a procedure of the "smoothing" type [9], in which the chain of equations connecting all the multipoint moments of the solu-

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 20–31, July–August, 1979. Original article submitted May 5, 1978.

 0021-8944/79/2004-0406$07.50

tion is broken by replacing the mean of the product of random functions with the product of their means.

The method can be used practically without change for investigating an electric field and solving analogous problems in the theory of elasticity for a medium with a large number of cracks [10].

1. We consider a conductive body containing a set of arbitrarily arranged cracks. By a crack we shall mean an infinitesimally thin cut along a segment of a smooth oriented surface with normal $\mathbf{n}$. The boundary condition on the surface of a crack is that the normal component of the electric current vector vanishes. Suppose that the dimensions of the cracks and the distances between them are much smaller than the dimensions of the body and the characteristic scale of variation of the external field (the field in the absence of any cracks). Then, disregarding boundary effects, we can immediately consider an infinite medium with cracks in a constant external field. Hereafter we shall assume for the sake of definiteness that we fix an external field of current vector j_0.

The solution of this problem can conveniently be sought in the form of the potential of a double layer concentrated on the surfaces of the cracks Ω_i:

$$\varphi(\mathbf{r})=\varphi_0+\sum_i \int \Phi(\mathbf{R}, \mathbf{n}_i')\, b_i(\mathbf{r}')\, \delta[\Omega_i]\, dr', \tag{1.1}$$

where $\mathbf{r}$ is the radius vector of a point of the medium; $\mathbf{R} = \mathbf{r}' - \mathbf{r}$; $\mathbf{n}' = \mathbf{n}(\mathbf{r}')$; $\delta[\Omega_i]$ is the delta function concentrated on the surface Ω_i [11]; φ_0 is the potential of the external field; $\Phi(\mathbf{R}, \mathbf{n})$ is the kernel of the potential of the double layer, which is the solution of the equation

$$\operatorname{div}[\mu\nabla\Phi] = \mathbf{n}\mu\nabla\delta(\mathbf{R}), \tag{1.2}$$

where μ is the electrical conductivity tensor; ∇ is the gradient operation; $\delta(\mathbf{R})$ is the Dirac delta function; tensor quantities standing one after another rotate by one index each.

If the potential densities $b_i(\mathbf{r})$ in (1.1) are known, the voltage of the electric field $\mathbf{E}$ and the current vector $\mathbf{j}$ in the medium can be represented in the form

$$\begin{aligned} \mathbf{E}(\mathbf{r})&=\mathbf{E}_0+\sum_i \int \nabla\Phi(\mathbf{R}, \mathbf{n}_i')\, b_i(\mathbf{r}')\, \delta[\Omega_i]\, dr', \\ \mathbf{j}(\mathbf{r})&=\mathbf{j}_0+\sum_i \int \mathbf{I}(\mathbf{R}, \mathbf{n}_i')\, b_i(\mathbf{r}')\, \delta[\Omega_i]\, dr', \end{aligned} \tag{1.3}$$

where $\mathbf{E}_0 = \nabla\varphi_0$; the kernel $\mathbf{I}(\mathbf{R}, \mathbf{n})$ has the form

$$\mathbf{I}(\mathbf{R}, \mathbf{n}) = \mu[\nabla\Phi - \mathbf{n}\delta(\mathbf{R})]. \tag{1.4}$$

As is known, the potential of the double layer (1.1) has discontinuities in the transition through surface Ω_i. Therefore the gradient of the field $\mathbf{E} = \nabla\varphi$ contains delta functions concentrated on these surfaces. However, the physical field of the current vector $\mathbf{j}(\mathbf{r})$ must be bounded everywhere except possibly at the edges of the cracks. The Dirac delta function appearing on the right side of (1.4) makes it possible to compensate for this singularity at the surfaces of the cracks. The vector field $\mathbf{j}(\mathbf{r})$, in the sense of generalized functions, satisfies the equation $\operatorname{div} \mathbf{j} = 0$ everywhere in the medium, as should in fact be the case, since there are no sources.

The equations for the function $b_i(\mathbf{r})$ follow from the boundary conditions at the surface of the cracks:

$$\mathbf{n}_i(\mathbf{r})\bar{\mathbf{j}}(\mathbf{r}) = 0 \quad \text{for} \quad \mathbf{r}\in\Omega_i \quad (i = 1, 2, \ldots).$$

The integral operators on the right sides of the relations (1.3) belong to the class of pseudodifferential operators [12], whose symbols (the Fourier transform of the kernels $\nabla\Phi$ and $\mathbf{I}$ with respect to the variable $\mathbf{R}$) are homogeneous functions of zero-degree homogeneity

$$\widetilde{\nabla\Phi}(\mathbf{k}, \mathbf{n}) = \mathbf{k}(\mathbf{kn})/k^2, \quad \tilde{\mathbf{I}}(\mathbf{k}, \mathbf{n}) = \mu[\mathbf{k}(\mathbf{kn})/k^2 - \mathbf{n}].$$

It is known [12] that such operators on the finite functions $f(\mathbf{r})$ admit the representations

$$\begin{aligned} \int \nabla\Phi(\mathbf{R}, \mathbf{n})\, f(\mathbf{r}')\, dr' &= \int \nabla\Phi(\mathbf{R}, \mathbf{n})\, f(\mathbf{r}')\, dr' + Af(\mathbf{r}), \\ \int \mathbf{I}(\mathbf{R}, \mathbf{n})\, f(\mathbf{r}')\, dr' &= \int \mathbf{I}(\mathbf{R}, \mathbf{n})\, f(\mathbf{r}')\, dr' + Bf(\mathbf{r}). \end{aligned} \tag{1.5}$$

Here the vector $\mathbf{n}$ is fixed, the integrals on the right are taken in the sense of the Cauchy principal value; the constants A and B have the form

$$A=\frac{1}{4\pi}\int\limits_{(\Gamma_1)}\nabla\widetilde{\Phi}(\mathbf{k},\mathbf{n})\,d\Gamma=\frac{1}{3}\mathbf{n},\quad B=\frac{1}{4\pi}\int\limits_{(\Gamma_1)}\widetilde{\mathbf{I}}(\mathbf{k},\mathbf{n})\,d\Gamma=\frac{2}{3}\mu\mathbf{n},$$

where Γ_1 is the surface of the unit sphere in the **k**-space of the Fourier transform.

In what follows, it will sometimes become necessary to define the action of operators with the kernels $\nabla\Phi$ and $\mathbf{I}$ on constants. For this purpose, we shall first consider a finite region V and select the boundary conditions in such a way that within V the current vector $\mathbf{j}$ is constant and equal to $\mathbf{j}_0$. Without changing the normal component of the current vector on the boundary, we introduce a number of cracks inside V. In a manner analogous to (1.3), the field of the current vector $\mathbf{j}(\mathbf{r})$ in the body can be represented in the form of potentials whose density is concentrated on the surfaces of the cracks Ω_i:

$$\mathbf{j}(\mathbf{r})=\mathbf{j}_0+\sum_i\int\mathbf{I}(\mathbf{r},\mathbf{r}',\mathbf{n}_i')\,b_i(\mathbf{r}')\,\delta[\Omega_i]\,dr',$$

where the kernel $\mathbf{I}$ has the form (1.4), and $\Phi(\mathbf{r},\mathbf{r}',\mathbf{n}')$ in the present case satisfies Eq. (1.2) and the boundary condition $\mathbf{m}\mu\nabla\Phi=0$ ($\mathbf{m}$ is the normal to the boundary of the body).

By virtue of the law of conservation, the mean integral value of the current in the region V remains equal to $\mathbf{j}_0$. Therefore

$$\left\langle\sum_i\int\mathbf{I}(\mathbf{r},\mathbf{r}',\mathbf{n}_i')\,b_i(\mathbf{r}')\,\delta[\Omega_i]\,dr'\right\rangle=0. \tag{1.6}$$

Here the pointed brackets represent averaging over the volume V. We increase the dimensions of the region V to infinity and assume that in this process the field of cracks becomes the realization of some random field of cracks of identical orientation which is homogeneous in space. In this passage to the limit the kernel $\mathbf{I}(\mathbf{r},\mathbf{r}',\mathbf{n}')$ tends to the kernel $\mathbf{I}(\mathbf{R},\mathbf{n}')$ for an infinite medium, and the mean integral value for an ergodic field of cracks can be replaced by the mean of the realizations over the ensemble. Then Eq. (1.6) takes the form

$$\int\mathbf{I}(\mathbf{R},\mathbf{n})\left\langle\sum_i b_i(\mathbf{r}')\,\delta[\Omega_i]\right\rangle dr'=0.$$

Since for a homogeneous field of cracks the mean under the integral sign is equal to a constant, we have the equation

$$\int\mathbf{I}(\mathbf{R},\mathbf{n})\,dr'=0. \tag{1.7}$$

From (1.6), (1.3) it follows that

$$\int\nabla\Phi(\mathbf{R},\mathbf{n})\,dr'=\mathbf{n}. \tag{1.8}$$

It is essential that the integrals appearing in these relations formally diverge at 0 and at infinity. Therefore formulas (1.7), (1.8) define some regularization of the divergent integrals on the left which, in general, is not unique. Indeed, if we fix not the external field of the current $\mathbf{j}_0$ but the voltage of the electric field $\mathbf{E}_0$, we arrive at regularizations of the form

$$\int\mathbf{I}(\mathbf{R},\mathbf{n})\,dr'=-\mu\mathbf{n},\quad\int\nabla\Phi(\mathbf{R},\mathbf{n})\,dr'=0.$$

It should be noted that there is no unique definition of the action of operators with kernels $\nabla\Phi$ and $\mathbf{I}$ on constants, and the values of the corresponding integrals are defined by the sense they have in the specific problem.

2. Now suppose that the set of plane elliptic cuts is the realization of some random field of cracks which is homogeneous in space. From this set we distinguish an arbitrary crack with surface Ω_i. If the potential densities $b_k(\mathbf{r})$ concentrated on the surfaces of all the cracks are known, then the current vector field $\bar{\mathbf{j}}_i(\mathbf{r})$ in which the specified crack is situated has the form

$$\bar{\mathbf{j}}_i(\mathbf{r})=\mathbf{j}_0+\sum_{k\neq i}\int\mathbf{I}(\mathbf{R},\mathbf{n}_k')\,b_k(\mathbf{r}')\,\delta[\Omega_k]\,dr',\quad\mathbf{r}\in\Omega_i. \tag{2.1}$$

The meaning of $\bar{\mathbf{j}}_i(\mathbf{r})$ is an external field for the cracks Ω_i, in which it behaves as though it were isolated. Hereafter we shall call $\bar{\mathbf{j}}_i(\mathbf{r})$ the effective external field of the crack Ω_i.

If the solution of the problem for an isolated crack in an arbitrary external field is known, i.e., if we know the explicit form of the function $b_k(\mathbf{r},\bar{\mathbf{j}}_k)$, then from (2.1) there follows a system of equations which is satisfied by the effective fields $\bar{\mathbf{j}}_i(\mathbf{r})$ for each of the interacting cracks

$$\bar{j}_i(\mathbf{r}) = \mathbf{j}_0 + \sum_{k\neq i} \int \mathbf{I}(\mathbf{R}, \mathbf{n}_k') b_k(\mathbf{r}', \bar{j}_k) \delta[\Omega_k] dr', \quad \mathbf{r} \in \Omega_i \ (i = 1, 2, \ldots).$$

To construct the statistical characteristics of the effective field, we introduce the following simplifying assumptions concerning its structure: a) The fields $\bar{j}_k(\mathbf{r})$ are practically constant for each of the cracks Ω_k but, in general, vary from crack to crack; b) the random field $\bar{j}_k$ is statistically independent of the dimensions and orientation of the crack Ω_k to which it relates.

The picture of the interaction between the cracks for which these hypotheses are realized can be qualitatively characterized as follows. For a typical crack the effective field (the sum of the external field and the field of all surrounding cracks) is approximately constant, and the contribution made to it by each individual crack is insignificant.

Without entering for the moment into a discussion of the region of applicability of these hypotheses, let us turn to an analysis of their formal consequences. From the solution of the problems for an isolated elliptical crack in a homogeneous external field j_k, we find that the function $b_k(\mathbf{r}, \bar{j}_k)$ has the form

$$b_k = h_k(\mathbf{r})(\mathbf{n}\bar{\mathbf{j}}_k), \tag{2.3}$$

where $h_k(\mathbf{r})$, in a cartesian coordinate system bound to the principal axes of the crack, can be represented in the form

$$h_k(x, y) = \frac{2c}{\mu E(\omega)} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{c^2}}. \tag{2.4}$$

Here a, c are the semiaxes of the ellipse; $E(\omega)$ is the complete elliptic integral of the second kind; $\omega = 1 - a^2/c^2$ ($c \geq a$); the medium is isotropic.

We introduce the generalized function $\delta[p(\mathbf{r})]$, concentrated on all the surfaces Ω_i:

$$\delta[p(\mathbf{r})] = \sum_i \delta[\Omega_i],$$

and the function $\delta_r[p(\mathbf{r}')]$, concentrated on the surfaces of all the cracks except for the one passing through the point $\mathbf{r}$,

$$\delta_r[p(\mathbf{r}')] = \sum_{k\neq i} \delta[\Omega_k] \quad \text{for } \mathbf{r} \in \Omega_i.$$

(The equation $p(\mathbf{r}) = 0$ gives the entire set of surfaces Ω_i.) Let $H(\mathbf{r})$ and $\mathbf{n}(\mathbf{r})$ be arbitrary continuous scalar and vector fields coinciding with $h_k(\mathbf{r})$ in (2.4) and $\mathbf{n}_k$ on the surfaces Ω_k. We consider the field $\bar{j}(\mathbf{r})$, defined at the points $\mathbf{r} \in \{p(\mathbf{r}) = 0\}$ (hereafter we shall write $\mathbf{r} \in p$) by the equation

$$\bar{\mathbf{j}}(\mathbf{r}) = \mathbf{j}_0 + \int \mathbf{I}(\mathbf{R}, \mathbf{n}') H(\mathbf{r}') [\mathbf{n}'\bar{\mathbf{j}}(\mathbf{r}')] \delta_r[p(\mathbf{r}')] dr' \quad (\mathbf{r} \in p). \tag{2.5}$$

If hypothesis a) is valid, it follows from (2.2), (2.3) that the field $\bar{j}(\mathbf{r})$ coincides with the field $\bar{j}_k$ on the surfaces of the cracks Ω_k. This field can be continued in an arbitrary manner to the entire space.

Starting with Eq. (1.5), we can construct the principal statistical characteristics of the effective field. We denote by $\bar{j}^n(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, the n-point moment of the vector field: the mean of the tensor product of the field $\bar{j}(\mathbf{r})$ at the points $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, provided that these points belong to the surfaces of the cracks. In particular, the mathematical expectation and the two-point moment of the effective field have the form

$$\bar{\mathbf{j}}^1 = \langle \bar{\mathbf{j}}(\mathbf{r}) | \mathbf{r} \in p \rangle, \quad \bar{\mathbf{j}}^2 = \langle \bar{\mathbf{j}}(\mathbf{r}_1) \otimes \bar{\mathbf{j}}(\mathbf{r}_2) | \mathbf{r}_1, \ \mathbf{r}_2 \in p \rangle.$$

Averaging both sides of Eq. (2.5) for the condition $\mathbf{r} \in p$ and making use of hypothesis b) that the field $\bar{j}(\mathbf{r})$ at the point $\mathbf{r}$ is statistically independent of the dimensions and orientation of the crack at that point, we obtain for $\bar{j}^1$ the expression

$$\bar{\mathbf{j}}^1 = \mathbf{j}_0 + \int \langle \mathbf{I}(\mathbf{R}, \mathbf{n}') \otimes \mathbf{n}' H(\mathbf{r}') \delta_r[p(\mathbf{r}')] | \mathbf{r} \in p \rangle \langle \bar{\mathbf{j}}(\mathbf{r}') | \mathbf{r}', \mathbf{r} \in p \rangle dr', \tag{2.6}$$

where the mean $\langle \bar{j}(\mathbf{r}') \ \mathbf{r}', \mathbf{r} \in p \rangle$ is calculated for the condition that the points $\mathbf{r}'$ and $\mathbf{r}$ are simultaneously situated on cracks. In defining the mean $\langle \bar{j}(\mathbf{r}') | \mathbf{r}', \mathbf{r} \in p \rangle$, we can again start with Eq. (2.5). Carrying out the appropriate averaging process and again making use of hypothesis b), we will have

$$\langle \bar{\mathbf{j}}(\mathbf{r}) | \mathbf{r}, \mathbf{r}_1 \in p \rangle = \mathbf{j}_0 + \int \langle \mathbf{I}(\mathbf{R}, \mathbf{n}') \otimes \mathbf{n}' H(\mathbf{r}') \delta_r[p(\mathbf{r}')] | \mathbf{r}, \mathbf{r}_1 \in p \rangle \langle \bar{\mathbf{j}}(\mathbf{r}') | \mathbf{r}', \mathbf{r}, \mathbf{r}_1 \in p \rangle dr'. \tag{2.7}$$

Equations (2.6), (2.7) are not closed, since their right sides contain mean values of the effective field which have been calculated for other conditions than the averages on the left sides.

The chain of equations that arises in this way can be interrupted by introducing additional assumption concerning the structure of the conditional means. The simplest method for obtaining an equation for $\bar{j}^1$ consists in taking

$$\langle \bar{j}(r') | r', r \in p \rangle = \langle \bar{j}(r') | r' \in p \rangle = \bar{j}^1. \tag{2.8}$$

This is equivalent to the usual assumption of the self-consistent field method that all the particles (cracks) are situated in an identical constant effective field [8].

From (2.6) and (2.8) we have

$$\bar{j}^1 = j_0 + \int \langle I(R, n') \otimes n' H(r') \delta_r [p(r')] | r \in p \rangle dr' \bar{j}^1. \tag{2.9}$$

Thus, we have reduced the problem to calculating the integral on the right side of this equation.

The following approximation for $\bar{j}^1$ can be obtained if in Eq. (2.7) we set

$$\langle \bar{j}(r') | r', r, r_1 \in p \rangle = \langle \bar{j}(r') | r', r \in p \rangle.$$

In the case of a homogeneous field of cracks the mean on the right depends only on the difference $r' - r$:

$$\langle \bar{j}(r') | r', r \in p \rangle = \theta(r' - r).$$

The function $\theta(r' - r)$ (the mean value of the effective field at the point r' on condition that there is a crack at the point r) characterizes the pairwise interaction in the system of interacting cracks. It is obvious that as $|r' - r| \to \infty$, this function will tend to the mean value of the effective field $\bar{j}^1$. The equation for $\theta(r)$ follows from (2.7) and has the form

$$\theta(r) = j_0 + \int \langle I(R, n') \otimes n' H(r') \delta_r [p(r')] | r, r_1 \in p \rangle \theta(r - r') dr'. \tag{2.10}$$

It should be noted that within the framework of the present scheme, the following approximations do not yield further corrections for $\bar{j}^1$. It can be shown that the equation for $\langle \bar{j}(r) | r, r_1, r_2 \in p \rangle$, as $|r_2| \to \infty$, becomes (2.10). Analogously, all the equations for the more complicated conditional means obtained by using assumptions of the type (2.8) reduce to (2.10) under the appropriate passage to the limit. To construct the second moment of the effective field, $\bar{j}^2$, we multply the values of the field $\bar{j}(r)$ (Eq. (2.5)) at the distinct points r_1 and r_2 and average the result for the condition $r_1, r_2 \in p$:

$$\bar{j}^2(r_1 - r_2) = j_0 \otimes \langle \bar{j}(r_2) | r_1, r_2 \in p \rangle + \int \langle I(R, n') [n' \bar{j}(r')] H(r') \delta_r [p(r')] \otimes \bar{j}(r_2) | r_1, r_2 \in p \rangle dr'. \tag{2.11}$$

The equation for the function $\bar{j}^2(R)$ can be obtained by "splitting" the mean value of (2.1), using hypothesis and assumptions of the type (2.8):

$$\langle \bar{j}(r_2) | r_1, r_2 \in p \rangle = \bar{j}^1; \langle \bar{j}(r') \otimes \bar{j}(r_2) | r', r_1, r_2 \in p \rangle = \langle \bar{j}(r') \otimes \bar{j}(r_2) | r', r_2 \in p \rangle = \bar{j}^2(r' - r_2).$$

From this and (2.11) we obtain

$$\bar{j}^2(r_1 - r_2) = j_0 \otimes \bar{j}^1 + \int \langle I(r_1 - r') \otimes n' H(r') \delta_r [p(r')] | r_1, r_2 \in p \rangle \bar{j}^2(r' - r_2) dr'. \tag{2.12}$$

3. Now we turn to the analysis of a specific stochastic model of a field of cracks in space – the Poisson model.

Suppose that in the bounded volume V there are N points such that the position of each point is uniformly distributed in V and is independent of the position of the other points. The points are the centers of elliptic cracks of random dimensions and orientations, and the corresponding joint distribution functions are assumed to be given. We arrive at a Poisson field of cracks if we let V and N approach infinity in such a way that $\lim (V/N) = V_0 < \infty$. Obviously, in this process there is no correlation between the positions of the cracks.

To determine the mean value of the effective $\bar{j}^1$, we turn to Eq. (2.9). In calculating the mean value under the integral sign in (2.9), we first carry out the averaging over all cracks which have a fixed orientation n' at the point r', and then over all possible orientations:

$$\langle I(R, n') \otimes n' H(r') \delta_r [p(r')] | r \in p \rangle = \langle I(R, n') \otimes n' H(r') \delta_r [p(r')] | r \in p \rangle \rangle_{n'}. \tag{3.1}$$

The conditional mean on the right can be represented in the form

$$\langle H(\mathbf{r}')\,\delta_r[p(\mathbf{r}')]\,|\,\mathbf{r}\in p\rangle = \frac{\langle \delta[p(\mathbf{r})]\,\delta_r[p(\mathbf{r}')]\,H(\mathbf{r}')\rangle}{\langle \delta[p(\mathbf{r})]\rangle}, \quad (3.2)$$

where the mean is calculated over all the cracks which have orientation $\mathbf{n}'$ at the point $\mathbf{r}'$. For homogeneous fields of cracks the mean value (3.2) is a function of the difference $\mathbf{r}-\mathbf{r}'$. If the points $\mathbf{r}$ and $\mathbf{r}'$ do not lie in a single plane which has $\mathbf{n}'$ as its normal, then, since the positions of the cracks are uncorrelated, we have

$$\langle \delta[p(\mathbf{r})]\delta_r[p(\mathbf{r}')]H(\mathbf{r}')\rangle = \langle \delta[p(\mathbf{r})]\rangle\langle \delta_r[p(\mathbf{r}')]H(\mathbf{r}')\rangle.$$

Replacing the mean over the ensemble by the mean over the volume for a typical realization, we obtain

$$\langle \delta_r[p(\mathbf{r}')]\,H(\mathbf{r}')\rangle = \lim_{V\to\infty}\frac{1}{V}\int_{(V)}\delta_r[p(\mathbf{r}')]\,H(\mathbf{r}')\,d\mathbf{r}' = \lim_{V,N\to\infty}\frac{1}{V}\sum_{i\neq k}^{N}\int h_i(\mathbf{r}')\,\delta[\Omega_i]\,d\mathbf{r}'.$$

Taking account of the expression for the function $h_i(\mathbf{r})$ (2.4), we will have

$$\langle \delta_r[p(\mathbf{r}')]\,H(\mathbf{r}')\rangle = \frac{\lambda}{\mu},\quad \lambda = \frac{4}{3}\frac{\pi}{V_0}\left\langle \frac{a^2c}{E(\omega)}\right\rangle, \quad (3.3)$$

where the mean on the right side of the expression for λ is calculated over all the cracks with fixed normal $\mathbf{n}'$.

If the points $\mathbf{r}$ and $\mathbf{r}'$ lie in a single plane with $\mathbf{n}'$ as its normal, the results of the averaging will be different, since $\delta_r[p(\mathbf{r}')] = 0$, when $\mathbf{r}$ and $\mathbf{r}'$ are on the same crack.

Therefore the mean (3.2) differs from a constant only at the points of the plane $(\mathbf{r}-\mathbf{r}')\mathbf{n}' = 0$. It can be shown that the value of the integral in (2.9) does not change if the mean (3.2) is considered constant everywhere.

If in (2.9) we interchange the order of averaging over orientations and integration, we arrive at the equation

$$\bar{\mathbf{j}}^1 = \mathbf{j}_0 + \left\langle \int \mathbf{I}(\mathbf{R},\mathbf{n}')\otimes\mathbf{n}'\frac{\lambda}{\mu}\,d\mathbf{r}'\right\rangle \bar{\mathbf{j}}^1.$$

By virtue of the regularization (1.7), the integral on the right vanishes, and consequently in the present case

$$\bar{\mathbf{j}}^1 = \mathbf{j}_0. \quad (3.4)$$

Now let us consider Eq. (2.10). As in (3.1), we average the integrand first over all realizations with fixed orientation of the cracks at the point $\mathbf{r}'$, and then over all orientations

$$\langle \mathbf{I}(\mathbf{R},\mathbf{n}')\otimes\mathbf{n}'\langle H(\mathbf{r}')\,\delta_r[p(\mathbf{r}')]\,|\,\mathbf{r},\mathbf{r}_1\in p\rangle\rangle_{\mathbf{n}'}.$$

The conditional mean in this expression can be represented as

$$\langle H(\mathbf{r}')\,\delta_r[p(\mathbf{r}')]\,|\,\mathbf{r},\mathbf{r}_1\in p\rangle = \frac{\langle H(\mathbf{r}')\,\delta_r[p(\mathbf{r}')]\,\delta_r[p(\mathbf{r}_1)]\,\delta[p(\mathbf{r})]\rangle}{\langle \delta[p(\mathbf{r})]\,\delta_r[p(\mathbf{r}_1)]\rangle}.$$

It can be shown that for a Poisson field of cracks, the desired mean, to within terms that vanish upon further integration, takes the form

$$\mu\langle H(\mathbf{r}')\,\delta_r[p(\mathbf{r}')]\,|\,\mathbf{r},\mathbf{r}_1\in p\rangle = \frac{1}{\pi}\left\langle \frac{a}{E(\omega)}J(\mathbf{r}'-\mathbf{r}_1)\delta[(\mathbf{r}_1-\mathbf{r}')\,\mathbf{n}']\right\rangle + \lambda, \quad (3.5)$$

where the averaging on the right is carried out over all cracks with orientation $\mathbf{n}'$; $\delta[(\mathbf{r}_1-\mathbf{r}')\mathbf{n}']$ is a delta function concentrated in the plane $(\mathbf{r}_1-\mathbf{r}')\mathbf{n}' = 0$. The first term on the right side of (3.5) is the contribution made to the mean by the realizations for which the points $\mathbf{r}$ and $\mathbf{r}'$ lie on a single crack. The function $J(\mathbf{R})$ has the symmetry of an ellipse with semiaxes a and c, and the affine transformation $\mathbf{C}$ carrying this ellipse into a disk of unit radius carries $J(\mathbf{R})$ into the spherically symmetric function $J'(\xi)$ ($\xi = |\mathbf{C}\mathbf{R}|$):

$$J'(\xi) = \begin{cases} a^2\displaystyle\int_{\frac{1}{2}\xi}^{1}\frac{(\zeta-\xi)\sqrt{\xi(2\zeta-\xi)}+\zeta^2\left[\arcsin\left(1-\dfrac{\xi}{\zeta}\right)+\dfrac{\pi}{2}\right]}{\sqrt{1-\zeta^2}}\,\zeta\,d\zeta & \text{for } \xi\leqslant 2,\\ 0 & \text{for } \xi > 2. \end{cases} \quad (3.6)$$

The second term on the right side in (3.5) is the contribution made to the mean by the realizations for which the points $\mathbf{r}'$ and $\mathbf{r}_1$ lie on different cracks. Substituting (3.5) into (2.10), we obtain the following expression for $\theta(\mathbf{r})$:

$$\theta(\mathbf{r}) = \mathbf{j}_0 + \mu^{-1}\int\left\langle \mathbf{I}(\mathbf{r}-\mathbf{r}', \mathbf{n}')\otimes\mathbf{n}'\left\langle\frac{a}{\pi E(\omega)}J(\mathbf{r}')\right\rangle\delta(\mathbf{r}'\mathbf{n}')\right\rangle_{n'}\theta(\mathbf{r}-\mathbf{r}')\,dr' + \mu^{-1}\int\langle\lambda\mathbf{I}(\mathbf{r}-\mathbf{r}')\otimes\mathbf{n}'\rangle\theta(\mathbf{r}-\mathbf{r}')\,dr'. \tag{3.7}$$

As was noted above, the mathematical expectation of the effective field $\bar{\mathbf{j}}^1$ coincides with the value of the function $\theta(\mathbf{r})$ at infinity. In the special case of a uniform distribution over the orientations of disk-shaped cracks, the function $\theta(\mathbf{r})$ is spherically symmetric, and its value at infinity can be calculated, omitting the construction of the exact solution of Eq. (3.7). In this case, using (1.5), we obtain

$$\bar{\mathbf{j}}^1 = \left[\mathbf{1} - \frac{1}{3}\langle\lambda\mathbf{n}\otimes\mathbf{n}\rangle\right]^{-1}\mathbf{j}_0, \tag{3.8}$$

where $\mathbf{1}$ is the unit bivalent tensor.

We now proceed to construct the correlation function of the effective field $\bar{\mathbf{j}}^2(\mathbf{r}_1-\mathbf{r}_2)$. We confine ourselves for the sake of simplicity to a field of cracks of identical orientation. Then we will be interested only in the projection of the vector $\bar{\mathbf{j}}$ onto $\mathbf{n}$, the common normal to all the cracks. We write

$$\langle\mathbf{n}\bar{\mathbf{j}}(\mathbf{r}_1)\mathbf{n}\bar{\mathbf{j}}(\mathbf{r}_2)|\mathbf{r}_1, \mathbf{r}_2\in p\rangle = \psi(\mathbf{r}_1-\mathbf{r}_2).$$

Here we assume that the points $\mathbf{r}_1$ and $\mathbf{r}_2$ lie on different cracks. If the points $\mathbf{r}_1$ and $\mathbf{r}_2$ always lie on the same crack, then, by hypothesis a), concerning the constancy of the effective field within the limits of each crack, and hypothesis b) of Sec. 2, we will have

$$\langle\mathbf{n}\bar{\mathbf{j}}(\mathbf{r}_1)\mathbf{n}\bar{\mathbf{j}}(\mathbf{r}_2)|\mathbf{r}_1, \mathbf{r}_2\in p\rangle = \langle[\mathbf{n}\bar{\mathbf{j}}(\mathbf{r})]^2\mathbf{r}\in p\rangle.$$

From (2.12) and the relations (3.4), (3.5) we obtain an equation for the function $\psi(\mathbf{r})$:

$$\psi(\mathbf{r}) = (\mathbf{n}\mathbf{j}_0)^2 + \mu^{-1}\int\mathbf{n}\mathbf{I}(\mathbf{r}-\mathbf{r}', \mathbf{n})\left\langle\frac{2a}{\pi E(\omega)}J(\mathbf{r}')\right\rangle\delta(\mathbf{r}'\mathbf{n})\,dr'\,\langle[\mathbf{n}\bar{\mathbf{j}}(\mathbf{r})]^2\,|\,\mathbf{r}\in p\rangle + \mu^{-1}\lambda\int\mathbf{n}\mathbf{I}(\mathbf{r}-\mathbf{r}', \mathbf{n})\psi(\mathbf{r}')\,dr'. \tag{3.9}$$

Averaging the expression for $[\mathbf{n}\bar{\mathbf{j}}(\mathbf{r})]^2$ on the condition that the point r lies on a crack, we obtain the following equation for a Poisson field of cracks:

$$\langle[\mathbf{n}\bar{\mathbf{j}}(\mathbf{r})]^2\,|\,\mathbf{r}\in p\rangle = (\mathbf{n}\mathbf{j}_0)^2 + \mu^{-1}\lambda\int\mathbf{n}\mathbf{I}(\mathbf{r}')\psi(\mathbf{r}')\,dr'. \tag{3.10}$$

Solving Eq. (3.9) by the Fourier transform method, we will have

$$\tilde{\psi}(\mathbf{k}) = (\mathbf{n}\mathbf{j}_0)^2(2\pi)^3\delta(\mathbf{k}) + [1-\lambda\mu^{-1}\mathbf{n}\tilde{\mathbf{I}}(\mathbf{k})]^{-1}\mu^{-1}\mathbf{n}\mathbf{I}(\mathbf{k})F(\mathbf{k})\langle[\mathbf{n}\bar{\mathbf{j}}(\mathbf{r})]^2|\mathbf{r}\in p\rangle, \tag{3.11}$$

where $F(\mathbf{k})$ is the Fourier transform of the function $\langle(2a/\pi\,E(\omega))\,\mathbf{I}(\mathbf{R})\,\delta[\mathbf{R}\mathbf{n}]\rangle$, defined by Eqs. (3.5) and (3.6). Here we have assumed that $\tilde{\mathbf{I}}(0) = 0$, by virtue of the regularization (1.7).

Furthermore, using the Parseval formula, we find from (3.10) that

$$\langle[\mathbf{n}\bar{\mathbf{j}}(\mathbf{r})]^2\,|\,\mathbf{r}\in p\rangle = (\mathbf{n}\mathbf{j}_0)^2 + \frac{\lambda}{(2\pi)^3}\int\left\{1-\lambda\left[\frac{(\mathbf{k}\mathbf{n})^2}{k^2}-1\right]\right\}^{-1}\left[\frac{(\mathbf{k}\mathbf{n})^2}{k^2}-1\right]^2F(\mathbf{k})\,d\mathbf{k}\,\langle[\mathbf{n}\bar{\mathbf{j}}(\mathbf{r})]^2\,|\,\mathbf{r}\in p\rangle.$$

Calculating the integral on the right side of this equation, we can find the expression for $\langle[\mathbf{n}\bar{\mathbf{j}}(\mathbf{r})]^2\mathbf{r}\,|\in p\rangle$:

$$\langle[\mathbf{n}\bar{\mathbf{j}}(\mathbf{r})]^2|\mathbf{r}\in p\rangle = \sqrt{1+\lambda}(\mathbf{n}\mathbf{j}_0)^2. \tag{3.12}$$

Thus, the right side of Eq. (3.11) is completely defined.

4. The results obtained in Sec. 3 enable us to pass to the determination of the mathematical expectation and correlation function of the random fields of the current and voltage vectors of an electric field in a medium with cracks.

We average the equation (1.3) for the vectors $\mathbf{j}$ and $\mathbf{E}$, making use of the effective-field assumption:

$$\langle\mathbf{j}\rangle = \mathbf{j}_0 + \left\langle\int\mathbf{I}(\mathbf{R}, \mathbf{n}')(\mathbf{n}'\bar{\mathbf{j}}^1)\langle H(\mathbf{r}')\,\delta[p(\mathbf{r}')]\rangle dr'\right\rangle_{n'},$$

$$\langle\mathbf{E}\rangle = \mathbf{E}_0 + \left\langle\int\nabla\Phi(\mathbf{R}, \mathbf{n}')(\mathbf{n}'\bar{\mathbf{j}}^1)\langle H(\mathbf{r}')\,\delta[p(\mathbf{r}')]\rangle\,dr'\right\rangle_{n'}.$$

Here the outer averaging is carried out over all the orientations, and the mean under the integral sign is calculated for a fixed value of the normal $\mathbf{n}'$.

Since for a Poisson field of cracks we have $\langle H(\mathbf{r}')\,\delta[p(\mathbf{r}')]\rangle = \lambda/\mu$, where λ has the form (3.3), making use of the regularizations (1.7), (1.8), we will have

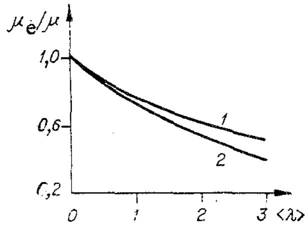

Fig. 1

$$\langle \mathbf{j} \rangle = \mathbf{j}_0, \ \langle \mathbf{E} \rangle = \mu^{-1}[\mathbf{j}_0 + \langle \lambda \mathbf{n} \otimes \mathbf{n} \rangle \overline{\mathbf{j}}^1], \tag{4.1}$$

where in the first approximation $\overline{\mathbf{j}}^1 = \mathbf{j}_0$; the second approximation for $\overline{\mathbf{j}}^1$ has the form (3.8).

We introduce the tensors of effective electrical resistance σ_e and effective electrical conductance μ_e by means of the natural relations

$$\langle \mathbf{E} \rangle = \sigma_e \langle \mathbf{j} \rangle, \quad \mu_e = \sigma_e^{-1}.$$

From this and (4.1) it follows that in the first approximation

$$\mu_e = \mu[1 + \langle \lambda \mathbf{n} \otimes \mathbf{n} \rangle]^{-1}. \tag{4.2}$$

The second approximation of the method yields

$$\mu_e = \mu\left[1 + \langle \lambda \mathbf{n} \otimes \mathbf{n} \rangle \left(1 - \frac{1}{3} \langle \lambda \mathbf{n} \otimes \mathbf{n} \rangle\right)^{-1}\right]^{-1}. \tag{4.3}$$

The relations (4.2) and (4.3), for the case of a uniform distribution of disk-shaped cracks over all orientations, is shown in Fig. 1 (curve 1 corresponds to (4.2), curve 2 to (4.3)).

We shall now derive an expression for the correlation function of a random field of a current vector in a medium with cracks of the same orientation. We multiply the value of the field $\mathbf{j}(\mathbf{r})$ (expression (1.2)) at the distinct points $\mathbf{r}_1$ and $\mathbf{r}_2$ and average the result over the ensemble of realizations. Making use of the effective-field assumptions we can obtain the following expression for the Fourier transform of the function $\mathbf{\Pi}(\mathbf{R}) = \langle \mathbf{j}(\mathbf{r}_1) \otimes \mathbf{j}(\mathbf{r}_2) \rangle$:

$$\tilde{\Pi}(\mathbf{k}) = (2\pi)^3 \delta(\mathbf{k})\, \mathbf{j}_0 \otimes \mathbf{j}_0 + \mu^2 \left[\frac{\mathbf{k}(\mathbf{k}\mathbf{n})}{k^2} - \mathbf{n}\right] \otimes \left[\frac{\mathbf{k}(\mathbf{k}\mathbf{n})}{k^2} - \mathbf{n}\right] \left[\tilde{K}(\mathbf{k}) \langle [\mathbf{n}\overline{\mathbf{j}}(\mathbf{r})]^2 \,|\, \mathbf{r} \in p \rangle + \lambda^2 \mu^{-2} \tilde{\psi}(\mathbf{k})\right].$$

Here $\tilde{\psi}(\mathbf{k})$ has the form (3.11); $\langle [\mathbf{n}\overline{\mathbf{j}}(\mathbf{r})]^2 \,|\, \mathbf{r} \in p \rangle$ is defined by the relation (3.12). The function $\mathbf{K}(\mathbf{R})$ is a mean value of the form

$$K(\mathbf{r}_1' - \mathbf{r}_2') = \langle H(\mathbf{r}_1') \delta_{r_1}[p(\mathbf{r}_1')] H(\mathbf{r}_2') \delta_{r_2}[p(\mathbf{r}_2')] \,|\, \mathbf{r}_1, \mathbf{r}_2 \in p \rangle,$$

where the averaging is carried out on the assumption that the points $\mathbf{r}_1'$ and $\mathbf{r}_2'$ lie on the same crack.

The mean energy density $\langle W \rangle$ of the field has the form

$$\langle W \rangle = \mu^{-1} \cdot \langle \mathbf{j}(\mathbf{r}) \otimes \mathbf{j}(\mathbf{r}) \rangle = \frac{1}{(2\pi)^3} \int \mu^{-1} \cdot \tilde{\Pi}(\mathbf{k})\, dk$$

(the dot indicates complete convolution of the tensors).

After calculating the integral on the right side of this relation, we obtain

$$\langle W \rangle = \mu^{-1}[j_0^2 + \lambda(\mathbf{n}\mathbf{j}_0)^2].$$

In an analogous manner, we can find an expression for the correlation function of the voltage vector of an electric field in a medium with cracks.

In conclusion, we shall consider the question of estimating the accuracy of the approximation of the effective field. It is known that in problems on the interaction of point particles, the more slowly the potential of an individual particle is attenuated at infinity and the higher is the particle density, the better will be the approximation yielded by this method. However, rigorous analytic estimates cannot be obtained in the case of strong interaction because of the complex structure of the exact solution. Usually such estimates are indicated on the basis of physical considerations.

With regard to the expressions obtained in this study for the first and second statistical moments of the solution, it should be noted that for a low concentration of cracks, when there is no interaction $(\overline{j}(\mathbf{r}) = j_0)$, hypotheses a) and b) of the method (Sec. 2) are satisfied exactly. Consequently the resulting expressions are exact solutions of the problem. As the concentration increases, hypotheses a) and b) will not, in general, be valid. Hypothesis a) will be violated when the effective field for a typical crack differs greatly from homogeneous field, i.e., when there is a high concentration of cracks. On the other hand, hypothesis b) is more justified when a typical crack lies in a field of many neighboring cracks, i.e., when there is a high concentration of cracks. This last remark also holds for the region of applicability of assumptions of the type (2.8).

However, the violation of hypotheses a) and b), which relate to the behavior of each individual crack, may not affect the value of such crude statistical characteristics as the first and second moments of the solution. Obviously, the effective-field method in the case of a Poisson field of cracks yields a good approximation for the first moments of the solution when the mean distance between the centers of the cracks is not less than their mean dimensions, which corresponds to $\lambda \lesssim 1.5-2$.

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